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Fuzzy Goals under Fuzzy Constraints

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GENERALITIES

In most economic theories, economic agents try to optimise an objective function. They are limited by physical, technological, or monetary constraints. Hence the consumer maximises his utility under a budgetary constraint and the producer maximises his profit or minimises his costs under technological constraints. The same problems exist in the case of fuzzy subsets. Thus Ponsard [5] has defined a fuzzy utility in the theory of the producer and the consumer. As in the case of ordinary subsets a decision problem boils down to optimising a fuzzy function under several fuzzy constraints.

The problem can be expressed as follows: given a nonfuzzy referential $\mathbf{X}$ and the membership functions $\mu_G(x)$, $\mu_{C_1}(x)$,..., $\mu_{C_n}(x)$ we have to determine x such that x maximises

$$\mu_D(x) = \mu_G(x) \cap \mu_{C_1}(x) \cap \cdots \cap \mu_{C_n}(x).$$

Negoita and Ralescu [3, 4] have shown that such a problem can come down to maximising a function on an ordinary subset. However, one drawback to this method is that in real life constraints do not play the same role. Therefore the problem can be brought down to studying the convex linear combinations of the membership functions of the fuzzy goal and the fuzzy constraints.

This paper will be divided into two parts: First, a theoretical part which presents the definition and existence of an optimum. Then a reciprocal will be built under more restrictive conditions; this procedure has the advantage of linking our study to that of Negoita and Ralescu. Second, the actual calculation of the optimum is defined in this way.

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1. Pareto Optimum

1.1. *Existence*

1.1.1. Theorem. *Given $(n+1)$ membership functions μ, $i \in [0, n]$ and $\alpha = (\alpha_0, ..., \alpha_n)$ a family of positive coefficients whose sum total equals one, any x^α maximising $\sum_i^n \alpha_i$ is such that*

(a) *if y exists such that $\forall i$, $\mu_i(y) > \mu_i(x^\alpha)$ then $y \notin \mathbf{X}$;*

(b) *if the α all differ from zero and if*

$$\forall i \in [0, n]\ \mu_i(y) \geqslant \mu_i(x^\alpha) \Rightarrow y \notin X$$

$$\exists j \in [0, n]/\mu_i(y) > \mu_i(x^\alpha) \Rightarrow y \notin X.$$

Proof. Let us proceed by reductio ad absurdum. $y \in \mathbf{X}$ exists such that $\mu_i(y) \geqslant \mu_i(x^\alpha)$ with at least one strict inequality. Let us multiply each inequality by α_i and add $\forall i\ \in (0, n)$ it follows

$$\sum_{i=0}^{n} \alpha_i \mu_i(y) > \sum_{i=0}^{n} \alpha_i \mu_i(x^\alpha)$$

which contradicts the definition of x^α.

By analogy with economics, x^α such as it is defined in part (a) is a weak Pareto optimum, in part (b) it is a strict Pareto optimum.

Before tackling the existence theorem let us give a definition.

1.1.2. Definition. x is unanimously preferred to y if and only if

$$\mu_i(x) \geqslant \mu_i(y) \qquad \forall i \in [0, n].$$

1.1.3. Existence Theorem. *If the functions μ_i are continuous $\forall i \in [0, n]$ if $\mathbf{X}$ $\mathbf{C}$ $\mathbb{R}^n$ is bounded $\forall y \in \mathbf{X}$, a Pareto optimum exists which is unanimously preferred to y.*

Proof. Let us examine the set of all the elements of $\mathbf{X}$ unanimously preferred to y.

$$\mathbf{Z}(y) = \{z, z \in \mathbf{X}\ \forall i \in [0, n], \mu_i(z) \geqslant \mu_i(y)\}$$

$$= \bigcap_{i=0}^{n} \{z_i z \in \mathbb{R}^m, \mu_i(z) \geqslant \mu_i(y)\} \cap X$$

$$Z(y) \neq \varnothing \qquad \text{for} \quad y \in \mathbf{Z}(y).$$

$\mathbf{Z}(y)$ is bounded, since $\mathbf{Z}(y)$ is a subset of $\mathbf{X}$ which is by hypothesis bounded. $\mathbf{Z}(y)$ is closed as intersection of closure. Hence $\mathbf{Z}(y)$ is compact.

The set of all the points of $\mathbf{Z}(y)$, where the continuous function μ_0 reaches its maximum is a non-empty compact:

$$\mathscr{X}_0 = \{x, x \in \mathbf{Z}(y)\ \forall z \in \mathbf{z}(y), \mu(x) \geqslant \mu(z)\}.$$

The operation is repeated by carrying on to $\mu_1,..., \mu_n$. We obtain $\mathscr{X}_n = \{x, x \in \mathscr{X}_{n-1}\ \forall z \in \mathscr{X}_{n-1},\ \mu_n(x) \geqslant \mu_n(z)\}$. Any element of $\mathscr{X}_n$ answers the question and we have

$$\mathscr{X}_n \mathbf{c} \mathscr{X}_{n-1} \subseteq ,..., \subseteq \mathscr{X}_0 \subseteq \mathbf{Z}(y).$$

This family is the only one possible. Indeed let us prove this by reductio ad absurdum. Let $t \in \mathbf{X}$ such that

$$\mu_i(t) \geqslant p_i(x) \qquad \forall i \in [0, n], x \in \mathbf{X}_n.$$

Therefore we have $\mu_i(t) \geqslant \mu_i(z)$, hence $t \in \mathbf{Z}(y)$. In particular for $i = 0$, $\mu_0(t) \geqslant \mu_0(x)$, $x \in X_n$ then $x \in \mathbf{X}_{0;}$ x maximises μ_0. On $\mathbf{Z}(y)$ we have μ_0 $(t) = \mu_0(x)$, $t \in X_0$.

The process is repeated and step by step we prove that $\forall i \in [0, n]$, $\mu_i(t) = \mu_i(x)$, therefore $t \in \mathscr{X}_n t$ is a Pareto optimum.

Remark. The basis hypothesis of the proof is $\mathbf{Z}(y)$ it is therefore possible to reduce the hypothesis by assuming than only x is compact.

The existence theorem is important because it provides a means of actually calculating the optimum. Before turning to the converse let us prove two lemmas.

1.2. *Converse*

1.2.1. LEMMA 1. *Let μ be the function defined on the convex set* $\mathbf{X}$ *towards R^{n+1},*

$$\mu = (\mu_0 \cdots \mu_n).$$

If the mappings μ are concave $\forall i \in (0, n)$, the set $\mu(X)$, R^{n+1} defined by

$$\mu(X)/\mathbb{R}^{n+1} = \{(v_0 \cdots v_n) \in \mathbb{R}^{n+1}; \exists x \in \mathbf{X}\ \mu_i(x) \geqslant v_i\}\ \forall i \in [0, n]$$

is convex.

Proof. Let $v = (v_0 \cdots v_n) \in \mu(\mathbf{X}) - R^{n+1}$ and $W = W_0,..., W_n) \in \mu(\mathbf{X}) - \mathbb{R}^{n+1}$ it suffices to show that $\forall t \in [0, 1]$ $tv + (1-t)^+ w \in \mu(\mathbf{X}) - R^{n+1}_+$:

(1) $v \in \mu(\mathbf{X}) - \mathbb{R}^{n+1}_+$ by definition $\exists x \in \mathbf{Y}/\mu i(x) \geqslant v_i\ \forall i \in [0, n]$

(2) $w \in \mu(\mathbf{X}) - \mathbb{R}^{n+1}_+$ by definition $\exists y \in \mathbf{X}/\mu_i(y) \geqslant w_i\ \forall i \in [0, n]$

Multiply (1) by t, (2) by $(1-t)$, whence $t\mu_i(x) \geqslant tv_i$,

$$(1-t)\,\mu_i(y) \geqslant (1-t)\,w_i \qquad \forall i \in [0, n].$$

Add member by member whence

$$tp_i(x) + (1-t)\,\mu_i(y) \geqslant tv_i + (1-t)\,w_i \qquad \forall i \in [0, n].$$

Now **X** is convex by hypothesis therefore $tx + (1-t)\,y \in \mathbf{X}$ the functions p_i are concave by hypothesis therefore

$$\mu_i(tx + (1-t)\,y) \geqslant t\mu_i(x) + (1-t)\,\mu_i(y).$$

Hence from the hypothesis and the preceeding inequalities it follows, $\mu_i(x + (1-t)\,y) \geqslant tv_i + (1-t)\,w_i$. Therefore $tv + (1-t)\;w \in \mu(X) - \mathbb{R}_+^{n+1}$ by definition of the set.

By definition $p_i(\mathbf{X}) - \mathbb{R}_+^{n+1}$ is convex.

1.2.2. LEMMA. *With the hypotheses of the preceding lemma, a necessary and sufficient condition for x to be a Pareto maximum is that*

$$(\mu(x) + \mathring{\mathbb{R}}_+^{n+1}) \cap (\mu(\mathbf{X}) - \mathbb{R}_+^{n+1}) = \varnothing.$$

Proof. Let us use a proof by reductio ad absurdum. Let us assume $(\mu(x) + \mathring{\mathbb{R}}_+^{n+1}) \cap (\mu(\mathbf{X}) - R^{n+1}) \neq \varnothing$, $y \in \mathbf{X}$, $p_i > 0$ $q_i \geqslant 0$ that $\forall i \in (0, n)$ exists, such that

$$\forall i \in (0, n), \qquad \mu_i(x) + p_i = \mu_i(y) - q_i.$$

Therefore $(y) > \mu_i(x)$ $\forall i \in (0, n)$ which contradicts the fact that x is Pareto optimum.

These two lemma will enable us to prove the converse of the existence of the Pareto optimum.

1.2.3. CONVERSE. *Let μ_i $i \in [0, n](n+1)$ be concave continuous membership functions. If $x \in \mathbf{X}$ is a weak Pareto optimum then a family $\alpha = (\alpha_0 \cdots \alpha_n)$ of positive coefficients whose sum equals one exists such that x maximises $\sum_{i=0}^n \alpha_i \mu_i$ on the set* **X**. *That is to say* $\forall\, y \in \mathbf{X}\, \sum_{i=0}^n \alpha_i \mu_i(x) \geqslant \sum_{i=0}^n \alpha_i \mu_i(y)$.

Proof. According to Lemma 1.2.1. the set $\mu(\mathbf{X}) - \mathbb{R}^{n+1}$ is convex. The set $\mu(x) + \mathbb{R}_+^{n+1}$ is the translate of the convex cone $\mathring{\mathbb{R}}_+^{n+1}$ by the vector $\mu(x)$ it is therefore convex.

According to Lemma 1.2.2. the two convex sets $\mu(X) - \mathbb{R}_+^{n+1}$ and $\mu(x) + \mathbb{R}^{n+1}$ are disjointed. According to Minkowski's separation theorem, a hyperplane exists which seperates them, i.e., a family of coefficients $a=(a_0,\ldots, a_n)$ and b exists such that

$$\forall v \in \mu(\mathbf{X}) - \mathbb{R}^{n+1} \left(\sum_{i=0}^{n} a_i v_i \right) \leqslant b \qquad \text{with} \quad v=(v_0,\ldots, v_n),$$

$$\forall w \in \mu(x+\mathring{\mathbb{R}}_{+}^{n+1}) \left(\sum_{i=0}^{n} a_i w_i \right) \geqslant b \qquad \text{with} \quad w=(w_0,\ldots, w_n),$$

$W \in \mu(x)+\mathring{\mathbb{R}}_{+}^{n+1}$ can be written

$$w_i = \mu_i + t_i, \qquad t=(t_0,\ldots, t_n),\ t>0.$$

Therefore $\sum_{i=0}^{n} a_i(\mu_i(x)+t_i) \geqslant b\ \forall t>0$. Let $\forall t>0$, $\sum_{i=0}^{n} a_i t_i \geqslant b - \sum_{i=0}^{n} a_i p(x)$.

As the functions are continuous by passing over the limit

$$0 \geqslant b - \sum_{i=0}^{n} a_i\ \mu(x).$$

That is to say

$$\sum_{i=0}^{n} a_i \mu_i(x) \geqslant b.$$

Moreover, we have the inequality

$$\left(\sum_{i=0}^{n} a_i c_i \right) \leqslant b$$

$v \in \mu(\mathbf{X}) - \mathbb{R}^{n+1}$, v can be written $v_i = \mu_i(y) + P_i$, $y \in \mathbf{X}$, $p_i \geqslant 0$, $\forall i \in (0, n)$.

Therefore $\forall y \in \mathbf{X}$, $\forall p_i \geqslant 0$, $\sum_{i=0}^{n} a_i(\mu_i(y) - p_i) \leqslant b$; in particular if $p_i = 0$ we have

$$\sum_{i=0}^{n} a_i \mu_i(y) \leqslant b.$$

By grouping the inequalities together we obtain $\forall y \in \mathbf{X}$,

$$\sum_{i=0}^{n} a_i \mu_i(y) \leqslant \sum_{i=0}^{n} a_i \mu_i(x).$$

All the coefficients are positive. Indeed let us prove this by reductio ad absurdum. Let us assume that $a_j < 0$ for a certain j. Let us go back to the inequality $\sum a_i t_i \geqslant b - \sum a_i \mu_i(x)$.

It holds true that $\forall t > 0$ by passing over the limit, it holds true $t=0$, $i \neq j_n$, $tj > 0$, the expression becomes $ajt \geqslant b - \sum_{i=0}^{n} a_i \mu_i(x)$.

If $t_j \to +\infty$, $a_j t\, j \to -\infty$ we arrive at an impossibility. Hence all the coefficients a_i are positive, in particular, $\sum a_i > 0$. The two members of the inequality can be divided by $\sum_{k=0}^{n} a_k$, it follows

$$\forall\, y \in \mathbf{X} \qquad \sum_{i=0}^{n} \frac{a_i}{\sum_{k=0}^{n} a_k} \mu_i(y) \leqslant \sum_{i=0}^{n} \frac{a_i}{\sum_{k=0}^{n} a_k} \mu_i(x).$$

It suffices to assume $a_i/\sum_{k=0}^{n} a_k = \alpha_0$ to obtain the desired result. Thus we have shown under an additional hypothesis the equivalance between the two methods of calculating the optimum.

In the second part we shall give a method for actually calculating the optimum.

2. Actual Calculation

2.1. *The setting of the Problem*

Let y be any element of $\mathbf{X}$ $\mu(y) = p_i i$, $\forall i \in [0, n]$. According to the existence theorem, the problem amounts to determining $x^0 \in \mathbf{X}$ such that x^0 maximises $\mu_0(x)$ under the constraints $\mu_i\ i\ (x^0) \geqslant \mu_i i$, $\forall i \in [1, n]$.

Such a problem is a problem of maximisation under an inequality constraint. Let us begin by giving some definitions.

2.1.1. Definitions. *Domain.* The set $\mathbf{D} = \{x \in \mathbf{X},\ \mu_i(x)(x) \geqslant \mu_i\}$ is called the domain of the problem.

Global maximum x^0 is a global maximum if and only if $x \in \mathbf{D}$ and $\forall\, y \in D$, $\mu_0(x) \geqslant \mu_0(y)$.

Local maximum x^0 is a local maximum if and only if $x \in \mathbf{D}$ and if a neighbourhood $\mathbf{V}$ of x^0 exists such that $\forall\, y \in \mathbf{V} \cap \mathbf{D}$, $\mu_0(x) \geqslant \mu_0(y)$.

By its very definition it is obvious that any global maximum is a local maximum.

By using particular fuzzy subsets we can prove a converse.

Fuzzy subset, (resp. strictly) *convex.* A fuzzy subset is (resp. strictly) convex if and only if its membership function is (resp. strictly) quasi-concave.

2.1.2. Theorem. *If the fuzzy subset* $(\mathbf{X}, \mu_i, 0)$ *is strictly convex and the fuzzy subsets* $(\mathbf{X}, \mu_i)$ $\forall i \in [1, n]$ *are convex, any local maximum is a global maximum.*

Proof. Let x^0 be a local maximum, let us use proof by reductio ad absurdum: $x \in \mathbf{D}$ exists such that

$$\mu_0(x) \geqslant \mu_0(x^0),$$

$x \in \mathbf{D}$; therefore $\mu_i(x) \geqslant \mu_i(y)$ $\forall i \in [1, n]$. Let us define $x' \in]x^0, x[$:

$$x' = x + t(x - x^0) = tx + (1 - t)x^0 \qquad t \in]0, 1[.$$

μ_i is quasi-concave therefore:

$$\mu_i(x) = \mu_i(tx + (1 - t)x^0) \geqslant \min[\mu_i(x), \mu_i(x^0)] \geqslant \mu_i$$

hence $x' \in D$.

We can choose t such that x belongs to a neighbourhood of x^0. Since x^0 is a local maximum $\mu_i(x') \leqslant \mu_0(x^0)$. By hypothesis $\mu_i(x) \geqslant \mu_0(x^0)$. Since μ_0 is strictly quasi-concave $\forall x' \in]x, x^0[$ we have $\mu_0(x') > \mu_0(x^0)$, which is impossible.

It is difficult to solve the problem such as it has been set thus we shall set the problem in two other ways and show their equivalence with the original problem.

2.2. *The Lagrangian Saddle Point*

2.2.1. Definition. Given the membership functions μ_i $i \in [0, n]$ defined on an open set X we determine $x^0 \in \mathbf{X}$, $y^0 \in \mathbb{R}^n$, $y_i \geqslant 0$ $\forall i \in [1, n]$ such that the couple (x^0, y^0) is a saddle point of the function $F(x, y) = \mu_0(x) + \sum_{i=0}^{n} y_i \mu_i(x)$, that is to say $\forall x \in \mathbf{X}$, $\forall y \in \mathbb{R}^n$, $F(x^0, y) \geqslant F(x^0, y^0) \geqslant F(x, y^0)$.

The function F is called a *Lagrange function or* lagrangian.

2.2.2. Theorem. *Any lagrangian saddle point is a global maximum for the function μ_0 under the constraint $\mu_i(x) \geqslant \mu_i$, $\forall i \in [1, n]$.*

Proof. Let (x^0, y^0) be a lagrangian saddle point. Let us show that $\forall i \in [1, n]$, $\mu_i(x^0) \geqslant 0$. Let us prove this by reductio ad absurdum, for example, $\mu_i(x^0) < 0$. Let us assume $z_i = y_1 + 1$, $z_i = y_i$ $\forall i \in [z, n]$:

$$F(x^0, z) = \mu_0(x^0) + (y_1 + 1)\, \mu_1(x^0) + \sum_{i=z}^{n} y_i \mu_i(x^0)$$

$$= \mu_0(x^0) + \sum_{i=1}^{n} y_i \mu_i(x^0) + \mu_1(x^0)$$

$$F(x^0, z^0) < F(x^0, y).$$

By construction $z \in \mathbb{R}^n$, $z_1 > 0$, $z_i \geqslant 0$, $\forall i \in [Z, n]$, $F(x^0, z) \leqslant F(x^0, z^0)$, we arrive at a contradiction. Thus $\mu(x^0) \geqslant 0$, $\forall i \in [1, n]$. Let us take $x \in \mathbf{D}$,

$$\mu_0(x) \leqslant \mu_0(x) + \sum_{i=1}^{n} y_i \mu_i(x) \qquad \text{for} \quad \mu_i(x) \geqslant 0,\ y_i \geqslant 0$$

$$\mu_0(x) \leqslant F(x, y^0) \leqslant F(x^0, y^0) \qquad \text{definition of } x^0 \text{ and } y^0$$

$$\leqslant F(x^0, 0) = \mu(x^0) \qquad \text{definition of } x^0$$

$$\mu_0(x^0) \geqslant \mu_0(x), \qquad x^0 \text{ is a global maximum.}$$

To prove the reciprocal we must use the preliminary lemma.

LEMMA 1. *Given a function f concave to n components defined on a convex $C \subseteq \mathbb{R}^m$ if the system $f_i(x) > 0$ has no solution then a function $f = \sum_{i=1}^n p_i f_i \ldots \sum_{i=1}^n p_i = 1$ exists such that $\forall x,\ \in C \sup f(x) \leqslant 0$.*

Proof. Let F be the set of points:

$$F = \{y_i,\ y \in \mathbb{R}^n\ \exists x \in C / f_i(x) > y_i\}.$$

The origin is not in F. Let us prove this by reductio ad absurdum $0_c \in F$ $\exists x \in C/f_i(x) > 0$, and by assumption, the system $f_i(x) < 0$ has no solution. F is convex $\forall y \in F$, $\forall y' \in F$, $\forall (p, p') \in \mathbb{R}^2$, $p \geqslant 0$ $p' \geqslant 0$, $p + p' = 1$ $py + p'y' \in F$. Indeed,

$$f_i(px + p'x') \geqslant pf_i(x) + pf_i(x') \qquad f \text{ concave,}$$

$$\geqslant py_i + p'y_i' \qquad \text{for} \quad f_i(x) > y_i,$$

$(py_i + p'y_i') \in F$ which is convex by definition.

By using Minkowski's separation theorem, there exist $\mathscr{A}_i$, $i \in [1, n]$, not all nul such that $\forall y \in F$, $\sum_{i=i}^n a_i y_i \leqslant 0$. Now if $x \in C$, we can define $\alpha_i < 0$ such that $y_i = f_i + \alpha_i$. Since $y \in F$ by definition $\sum_{i=i}^n a_i(f_i(x) + \alpha_i) \leqslant 0\ \forall \alpha_i < 0\ \sum_{i=1}^n a_i f_i(x) + \sum_{i=1}^n a_i \alpha_i \leqslant 0$. The function thus defined is a continuous function of α_i. We can write

$$\lim_{\alpha \to \infty} \left[\sum_{i=i}^{n} a_i f_i(x) + \sum_{i=i}^{n} a_i \alpha_i \right].$$

All the $\mathscr{A}_i$ are non-negative. Let us use prove this by reductio ad adsurdum: $\exists i/a_i < 0$, $\sum_{i=i}^n a_i' f_i(x) + \alpha_i) \leqslant 0$.

Let us make $\alpha_i \to -\infty \alpha_i - \alpha_{i-i} \alpha_n$ tends towards zero. The expression tends towards $+\infty$ which is impossible since it is negative or nul.

Let us assume that $p_k = a_k / \sum_{i=1}^n a_i$, $p_k \geqslant 0\ \forall k \in [1, n]$:

$$\sum_{k=1}^{n} p_k = \sum_{k=1}^{n} \left(a_k \Big/ \sum_{i=1}^{n} a_i \right) = 1.$$

The expression becomes $\forall x \in C$, $\sum_{i=1}^{n} p_i f_i(x) \leqslant 0$ if $n \leqslant m+1$; the theorem is proved to hold true. If $n \leqslant m+1$ we can cancel at least $n-(m+1)$ of the p_i coefficients. Let us assume $C_i = \{x: x \in C f_i(x) > 0\}$ $\forall i \in [1, n]$. By assumption, the system $f_i(x) > 0$ has no solution hence $\bigcap_{i-i}^{n} C_i = \varnothing$; according th Helly's theorem the indices $i_1, ..., i_{m+1}$ exists such that the intersection of the sets $C_{i1}, ..., C_{im+1}$ is empty. The system $f i_1(x) > 0, f i_{m+i}(x) > 0$ has no solution so we return to the previous case in which $n = m+1$.

CONSEQUENCE 1. *Given a convex set C and a family f_i of upper semi-continuous concave functions in C, if the finite or non-finite system $f_i(x) > 0$ $\forall i \in [1, n]$ has no solution in C then a function f exists such that $f = \sum_{i=1}^{m+1} p_i f_i$, $p_i \geqslant 0$, $\forall i \in [1, m, +1]$:*

$$\sum_{i=1}^{m+1} p_i = 1 \qquad \forall x \in C, \sup f(x) < 0.$$

Proof. Let $fi(x) > \varepsilon_i$ be the system which has no solution in C; the sets $C_i = \{x: x \in C, f_i(x) \geqslant \varepsilon_i\}$ are closed and their intersection is empty. Therefore the system $f_i(x) - \varepsilon_i > 0$ $\forall i \in [i, n]$ has no solution in C; we can apply the preceeding theorem. There exists $p_i \geqslant 0$, $\forall i \in [1, m+1]$, $\sum p_i = 1$ such that $p_i(f_i(x) - \sum i\varepsilon) \leqslant 0$, whence $\sum_{i=1}^{m+1} p_i f_i(x) \leqslant \sum_{i=1}^{m+1} p_i \varepsilon_i < 0$. Now the function f is upper semi-continuous on the compact C: by using the properties of the bounds we can write

$$\sup_{x \in C} f(x) = \sup_{x \in C} \sum_{i=1}^{m+1} p_i f_i(x) \leqslant \sum p_i \varepsilon_i < 0.$$

LEMMA 2. *Let $A \in \mathbb{R}^n$ and $BC \subseteq \mathbb{R}^n$ be the compact convex sets, let $E(x, y)$ be a function of $R^m R^n$ towards R, which is concave upper semi-continuous relative to x convex semi-continuous relative to y a couple $(x^0, y^0) \in A \times B$ exists such that*

$$F_{x \in A}(x, y^0) \leqslant F(x^0, y^0) \leqslant F_{y \in B}(x^0, y).$$

Proof. Let $g(y) = \max_{x \in A} F(x, y)$. G is a function of y and is defined as lower semi-continuous on the compact B it has a lower bound and actually reaches it. There exists $y^0 \in B / g(y^0) = \min_{y \in B}[\max_{x \in A} F(x, y]$.

Let the function $h(x) = \min_{y \in B} F(x, y)$. As a function of x upper semi-continuous on the compact A, it has a maximum and actually reaches it; there exists

$$x^0 \in A / h(x^0) = \max_{x \in A} [\min_{y \in B} F(x, y)].$$

Let us show that $g(y^0) = h(x^0)$:

$$g(y) - h(x) = \max_{x' \in A} F(x', y) - \min_{y' \in B} F(x, y')$$

$$= \max [F_{x' \in A}(x', y) - F_{y' \in B}(x, y')] \geqslant 0$$

$$g(y) - h(x) \geqslant 0 \qquad F(x', y') - F(x', y') = 0.$$

Then

$$g(y^0) \geqslant h(x^0).$$

It remains to prove that $g(y^0) \leqslant h(x^0)$. Let us consider $hy(x) = F(x, y) - h(x^0) - \varepsilon$. The system $hy(x) \geqslant 0$ has no solution; let us prove this reductio ad absurdum that x' exists such that

$$\forall\, y \in B, \qquad hy(x) = F(x', y) - h(x^0) - \varepsilon \geqslant 0$$

$$F(x', y) \geqslant h(x^0) + \varepsilon > h(x^0).$$

In particular, $\min_{y \in B} F(x', y) > h(x^0)$ which contradicts the fact that $h(x^0)$ is a maximum in x.

Let us use the consequence of Lemma 1: a function $h(x) = \sum p_i h y_i(x)$, $r = n + m + 1$ exists verifying $h(x) < 0$. According to the definition of h,

$$h(x) = \sum_{i=1}^{r} p_i h y_i(x) \sum_{i=1}^{r} p_i F(x - y_i) - h(x_0) - \varepsilon < 0$$

$$\sum_{i=1}^{r} p_i(x_i y_i) < h(x_0) + \varepsilon).$$

Now F is concave relative to y:

$$F(x, y) = F\left(x_i \sum_{i=1}^{r} p_i y_i\right) \leqslant \sum_{i=1}^{r} p_i F(x_i y_i).$$

Finally

$$F(x, y) < h(x_0) + \varepsilon$$

$$\max_{x \in A} F(x, y) < h(x_0) + \varepsilon$$

$$g(y) < h(x_0) + \varepsilon$$

$$g(y_0) < h(x_0) + \varepsilon$$

The relation holds true whatever the value of ε, in particular when $\varepsilon \to 0$; from the properties of the bound $g(y_0) \leqslant h(x_0)$ which brings the proof to an end.

2.2.2. CONVERSE. *If the functions $\mu_0, \ldots, \mu_n$ are concave if a point x exists verifying the constraints that $p_i(x) \neq 0$ for all the constraints which are not linear affine, if x^0 is the maximum of μ_0, $y^0 \in \mathbb{R}^n$, $y^0 \geqslant 0$ exists such that (x^0, y^0); that is to say, a lagrangian saddle point $F(x, y) = \mu_0(x) + \sum y_i(\mu_i(x) - \mu_i)$.*

Proof. The system $\mu_i(x) - \mu_i > 0$ $\forall i \in [1, n]$, $\mu_0(x) - \mu_0(x^0) > 0$ has no solution because x^0 is maximum. Let us apply Lemma 3. y^0 exists such that $y_i^0 \geqslant 0$ $\forall i \in [1, n]$ such that $\forall x \in \mathbf{X}$,

$$\mu_0(x) - \mu_0(x^0) + \sum_{i=1}^{n} y_i^0(\mu_i(x) - \mu_i) \leqslant 0,$$

$\forall x \in \mathbf{X}$, $\mu_0(x) \leqslant {}_0(x^0) + \sum_{i=1}^n y_i(\mu_i(x) - \mu_i)$:

$$F(x, y^0) \leqslant \mu_0(x^0).$$

Similarly, $\mu_0(x^0) \leqslant \mu_0(x^0) + \sum_{i=1}^n y_i(\mu_i(x^0) - \mu_i)$ for $\mu_i(x^0) - \mu_i) \geqslant 0$ whence $\forall y \in \mathbb{R}_+^n$, $f(x^0) \leqslant F(x^0, y)$. By bringing together all these conditions,

$$F(x, y^0) \leqslant \mu_0(x^0) \leqslant F(x^0, y).$$

If $x = x^0$, $y = y^0$, $\mu_0(x^0) = F(x^0, y^0)$, we have $\forall x \in \mathbb{R}^m$, $\forall y \in \mathbb{R}^n$,

$$F(x, y^0) \leqslant F(x^0, y^0) \leqslant F(x^0, y).$$

Thus we have shown that the two methods are equivalent: however, the analysis of the lagrangian saddle point is complicated so in order to get to the Lagrange problem we shall assume that the membership functions can be differentiated.

2.3. *Lagrange Problem*

2.3.1. Given the functions μ_i, $i \in [0, n]$, which are defined, continuous, and derivable on an open set $\mathbf{X}$, the Lagrange problem is the determination of $x^0 \in X$, $y^0 \in \mathbb{R}^n$, $y_i \geqslant 0$, such that $y^0 \mu_i(x^0) = 0$, $\forall i \in [1, n]$:

$$\text{grad } F(x^0, y^0) \leqslant \text{grad}\left[\mu_0(x^0) + \sum_{i=1}^{n} y_i(\mu_i(x^0) - \mu_i\right] = 0.$$

2.3.2. THEOREM. *Any lagrangian saddle point is a solution to the Langrange problem.*

Proof. Let (x^0, y^0) be a lagrangian saddle point. We can, as previously, prove that $\mu_i(x^0) > 0$, $\forall i \in [1, n]$.

Moreover $F(x^0, y) = \mu_0(x^0) + \sum_{i=1}^{n} y_i(\mu_i(x^0) - \mu_i)$. Now $r(x^0, y)$ is a minimum of $F(x^0, y)$. Since $\sum_{i=1}^{n} y_i(\mu_i(x^0) - \mu_i) \geqslant 0$. This condition can only be fulfilled if $\sum_{i=1}^{n} y_i(x^0) - \mu_i) = 0$. Since all the lemmas of ε are positive or nil, this condition is fulfilled if $y_i^0\, \mu_i(x^0) = 0$, $\forall i \in [1, n]$.

Furthermore $\forall x \in \mathbf{X}$, $F(x^0, y^0)$ is an extremmum, so grad $F(x^0, y^0)$ must equal 0.

2.3.3. THEOREM. *Given μ_i, $i \in [0, n]$ of the continuously derivable functions of a convex set* $\mathbf{X}$, *if μ_0 is quasi concave, if μ_i, $i \in [1, n]$, are strictly quasi-concave, any solution to the Lagrange problem is a maximum of μ_0 under the constraints.*

Proof. Let $I = \{i \in [1, n], \mu_i(x^0) = \mu_i\}$ $\forall i \notin I$, $\mu_i(x^0) \notin \mu_i)$, *and* $y_i^0 = 0$.

Let $i \in I$, $\forall x \in \mathbf{D}$, by definition $\mu_i(x^0) - \mu_i > 0$ therefore $\mu_i(x) > \mu_i < \mu_i(x^0)$. Now μ_0 is strictly quasi-concave, therefore $\forall i \in I$.

$$\langle x - x^0, \text{grad } \mu(x^0) \rangle \geqslant 0.$$

Let us use the scalar product properties:

$$\begin{array}{lll} \forall i \in I, & y_i \langle x = x^0, \text{grad } \mu_i(x^0) \rangle = \langle x - x^0, \text{grad } y_i \mu_i(x^0) \rangle \geqslant 0 \\ \forall i \notin I, & y_i = \qquad\qquad\qquad\qquad 0. \end{array}$$

Whence

$$\langle x - x^0, \text{grad } y_i \mu_i(x^0) \rangle \geqslant 0$$

$$\left\langle x - x^0, \text{grad } \sum_{i=1}^{n} y_i \mu_i(x^0) \right\rangle.$$

Now grad$(\mu(x^0) + \sum_{i=1}^{n} y_i \mu_i(x^0)$.
Whence

$$\left\langle x - x^0, \text{grad}(\mu_0(x^0) + \sum_{i=1}^{n} y_i \mu_i(x^0) \right\rangle = 0,$$

$$\langle x - x^0, \text{grad } \mu_0(x^0) \rangle > 0,$$

$$\left\langle x - x^0, \text{grad } \sum_{i=1}^{n} y_i \mu_i(x^0) \rangle \right\rangle 0,$$

$$\langle x - x^0, \text{grad } \mu_0(x^0) \rangle \geqslant 0.$$

μ_0 is quasi-concave, $\mu_0(x) \leqslant {}_0(x^0)$ $\mu_0(x^0)$ is maximum.

LEMMA 3. *Let* $f g_i$, $i \in [i, n]$, *be concave functions defined on* $\mathbb{R}^n$ *such that the* $n-q$ *last equations are affine if the system* (1) $g_i(x) \geqslant 0$, $\forall i \in [i, n]$, $f(x) > 0$ *has no solution in* $\mathbb{R}^n$ *and if the system* (2), $g_i(x) > 0$ $\forall i \in [1, q]$, *and* $g_i(x) \geqslant 0$ $\forall i \in [q+1, n]$, *has a solution there exist* $y_i \geqslant 0$, $i \in [1, n]$, *not all will such that* $\forall x \in \mathbb{R}^n$,

$$f + \sum_{i=1}^{n} y_i g_i \leqslant 0.$$

Proof. From Lemma 1, a function $h = p_0 f + \sum p_i y_i$ exists such that $\forall x \in \mathbb{R}^m$ $h \leqslant 0$. If $p_0 \neq 0$ let us assume $p_0 y_i = p_i$; we have $h = p_0[f + \varepsilon ...]$, thus $f = \sum y_i g_i \leqslant 0$ and the theorem is proved to hold true.

If $p_0 = 0$ we use a proof by reductio ad absurdum. Let x^0 be a solution to system (2) therefore $g(x^0) \geqslant 0$. Now according to the definition of h, $h = \sum y_i g_i \leqslant 0$. There remains the condition $h(x^0) = 0$, $\sum y_i g_i(x^0)$ $\forall i \in [1, q]$, $g_i(x^0) > 0$. For the condition to be filled p_i must equal 0, $\forall i \in [1, q]$.

Finally, $h(x) = \sum_{i=q+i}^{n} y_i g_i(x^0)$; h is an affine function of x which cancels itself out for $x = x^0$ and which is constantly negative or nul. Therefore for the condition to be fulfilled $\forall x \in \mathbb{R}^m$, $h(x) = 0$.

Let us assume that we remove any one of the inequalities of system (1); it then becomes compatible. Let us introduce the $m-q$ closed convex sets $C_i = \{x_i x \in R^m\, g_i(x) \geqslant 0 \quad \forall i \in (q_i)$ and the convex set $C = \{x; x \in \mathbb{R}^m\, f(x) > 0; g_i(x) \geqslant 0\ \forall i \in [1, q]$, $Ci \cap C \neq 0$ becauxe system (1) has no solution.

But the intersection of any $m-q-1$ among them with the convex C is not empty; it follows, according to Helly's theorem, that C is not part of their union; $x' \notin C$ exists such that $x^1 \in \bigcup_{i=q+1}^{n} C_i$. Then $g_i(x^1) < 0$, $\forall i \in [q+1, m]$,

$$h(x^1) = \sum_{i=q+1}^{n} p_i g_i(x^1) < 0 \qquad \text{which is impossible.}$$

According to the existence theorem in order to continue the calculation we have to solve the program

$$\mu_1(x).$$

Under the constraints

$$(x) \geqslant \mu_i(x^0) \qquad \forall i \in [2, n],$$

x being the solutions to (1): the program and the procedure is repeated.

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